# Melzer's identities revisited

## Omar Foda and Trevor A. Welsh

*Dedicated to Professor George Andrews on the occasion of his 60th birthday.*

ABSTRACT. We further develop the finite length path generating transforms introduced previously, and use them to obtain constant sign polynomial expressions that reduce, in the limit of infinite path lengths, to parafermion and ABF Virasoro characters. This provides us, in the ABF case, with combinatorial proofs of Melzer's boson-fermion polynomial identities.

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## 1. Introduction

One can think of exactly solvable models, in statistical mechanics [**5**] and in quantum field theory [**13**], as concrete realisations of certain mathematical structures. These structures are so powerful, that they allow us to compute, at least in principle, an infinite number of physical quantities in each solvable model. Computing *one* such quantity suffices to call the corresponding model *solved*.

Of particular interest are the connections between exact solutions and infinite dimensional algebras [**18**]. One aspect of this connection is the observation, first made in [**14**], that the one-point functions of regime-III restricted solid-on-solid ABF models [**3**], suitably normalised, turn out to be characters of Virasoro highest weight modules [**8**][1].

Similarly, the one-point functions of regime-II ABF models are characters of parafermion highest weight modules[2].

The purpose of this work is to discuss combinatorial aspects of the ABF and parafermion one-point functions, or equivalently the corresponding Virasoro and parafermion characters[3]. The shortest route to the combinatorics that we are interested in is *via* the statistical mechanical side of the problem.

From the lattice point of view, a one-point function is the normalised generating function of an infinite set of two-dimensional configurations with very complicated weights (typically products of trigonometric or even elliptic functions). Baxter's corner transfer matrix method reduces the above problem to computing the generating function of an infinite set of one-dimensional configurations with relatively very simple weights (typically simple powers of a parameter $q$) [**5**].

From a combinatorial point of view, the set of weighted one-dimensional configurations is the starting point of this work. One does not need to know anything about the underlying physical models, or their connections with infinite dimensional algebras[4]. We are handed a set of one-dimensional combinatorial objects, rules for computing their weights, and the task of computing their generating functions[5].

There is no unique method to compute the generating function of a weighted set. Different methods produce different expressions. Since they all represent the same generating function, equating them produces $q$-series identities.

One way to compute a generating function is 'sieving', or inclusion-exclusion [**4**]. By construction, this method produces an expression whose terms have alternating signs. In other words, the coefficients of the $q$-series so expressed, are not manifestly positive definite. However, we know that they are positive definite,

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[1]For the rest of this work, we refer to the regime-III restricted solid-on-solid ABF models simply as 'ABF models'. The spectrum generating algebra of the ABF models is the Virasoro algebra of [**16**], with central charge $c = 1 - 6/p(p+1), p = 3, 4, \cdots$

[2]For the rest of this work, we refer to regime-II restricted solid-on-solid ABF models simply as 'parafermion models'. The spectrum generating algebra of the parafermion models is the $\mathbb{Z}$ algebra of [**19**], with central charge $c = 2(p-2)/(p+1), p = 3, 4, \cdots$

[3]As we will see below, the ABF and parafermion models are related by the transformation $q \to q^{-1}$, where $q$ is the nome of the elliptic functions used to parametrise the two-dimensional weights of these models, or equivalently, the expansion parameter that appears in the $q$-series expression of the characters. For that reason, for each statement that we make about the ABF models, a corresponding statement can be made about the parafermion models.

[4]We refer the reader to [**18**] for an excellent introduction to the algebraic approach to exactly solvable lattice models.

[5]We refer to computing the generating function of a set of weighted object simply as *q-counting*.

since we are counting objects. The alternating-sign $q$-series expressions for the one-dimensional configurations coincide with the Rocha-Caridi expressions for the Virasoro characters [21].

In the context of the Virasoro characters, the Stony Brook group were the first to conjecture that there exist constant-sign $q$-series expressions[6]. For physical reasons that are beyond the scope of this work, the alternating-sign expressions are referred to as *bosonic*. The constant-sign expressions are referred to as *fermionic*. On equating these expressions, one obtains *boson-fermion $q$-series identities*.

One approach to proving such identities, is to work at the level of finite versions of the combinatorial objects under consideration. $q$-Counting these finite sets produces boson-fermion $q$-polynomial identities. Since the initial conjectures of the Stony Brook group, there has been many further conjectures and proofs of $q$-polynomial identities. In this work, we are interested in Melzer's polynomial identities [20]. For each one-point function, of each ABF model, Melzer conjectured four boson-fermion $q$-polynomial identities[7]. Because of their relative simplicity, these conjectures have served as ideal testing grounds for various approaches towards proving boson-fermion identities. Proofs of a subset of Melzer's identities were obtained in [**6, 11, 10**] using recursion techniques. A complete proof using the same methods is given in [**22**]. Combinatorial proofs of a subset of these identities were obtained in [**12, 17, 24, 25**].

In this work, we are interested in a combinatorial proof of the full set of Melzer's identities. We obtain such a proof by extending our previous work on path generating transforms [**15**]. Though, strictly speaking, we do not obtain new final results, the method that we use is new. We hope that this method gives further insight into the combinatorics of Virasoro highest weight modules, which have turned out to be such rich and fascinating objects.

**1.1. Outline of paper.** In Section 2.1, we define the combinatorial objects, called *paths*, that we are interested in $q$-counting. In sections 2.2 and 2.3, we define, in terms of paths, the two generating functions of most importance to us[8].

The first, $X_L^{p'}(a, b, c)$, gives the finitised one-point function of the ABF models. The second, $x_L^{p'}(a, b, c)$, gives the finitised one-point function of the parafermion models. Lemma 2.4 states the relationship between $X_L^{p'}(a, b, c)$ and $x_L^{p'}(a, b, c)$[9].

However, instead of the two functions $X_L^{p'}(a, b, c)$ and $x_L^{p'}(a, b, c)$, we prefer to work with certain renormalisations thereof, which we denote $\chi_{a,b,c}^{p'-1,p'}(L)$ and $\chi_{a,b,c}^{1,p'}(L)$ respectively[10]. We make this change for two reasons. Firstly, the analysis here then parallels that of [**15**] and a comparison can be readily made. Moreover, the techniques of this work and those of [**15**] are then ready to combine so that the other cases of [**9**] may be investigated. Secondly, the connection with partitions

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[6]For a complete listing of the papers of the Stony Brook group on this subject, we refer the reader to [**23**].

[7]For each one-point function, these identities are not independent. As we will see below, two of them are related to another two, that belong to another one-point function, by means of a simple up-down reflection of the combinatoric objects that are counted.

[8]In [**3**], these generating functions are denoted $X_L(a, b, c)$. and $x_L(a, b, c)$.

[9]In [**3**], $x_L^{p'}(a, b, c)$ is defined through its relationship with $X_L^{p'}(a, b, c)$, and not directly in terms of the paths as we do here.

[10]Here, we retain the notation of [**15**].

satisfying prescribed hook-difference conditions, analysed in [2], is then apparent. We discuss this in Appendix B, relying somewhat on the analysis of [15].

Section 2.4 indicates how $\chi_{a,b,c}^{1,p'}(L)$ may be directly determined from the paths. Our strategy is to obtain expressions for $\chi_{a,b,c}^{1,p'}(L)$ using combinatorial techniques applied to the paths, and then to obtain expressions for the other functions mentioned above from these. We choose to work with these parafermion models for compatibility with our work in [15]. Furthermore, combining the techniques with those of [15] will, in future work, enable further models from [9] to be tackled.

In Section 2.4, each vertex of a path is designated either *scoring* or *non-scoring*. In Section 2.5, the *striking sequence*[11] of a path is defined, and the means to designate the first point and last point of a path as scoring or non-scoring is given. In Section 2.6, we define the generating function $\chi_{a,b,e,f}^{1,p'}(L,m)$ for paths having a certain length, a certain number of non-scoring vertices, and first and last vertices of a certain nature.

In Section 3, we introduce the cornerstone of our method. This is the notion of a transform which enables us to express the generating functions in terms of those of a 'simpler' model. This transform is called a $\mathcal{B}$-transform.

Following the action of a $\mathcal{B}$-transform, the path may be extended by adding a number of segments, alternating in direction, to the left end. This process, which we refer to (for physical reasons that do not concern us here) as *inserting particles*, is described in Section 3.2. These particles are then allowed to *move* through the path, as described in Section 3.3[12]. As shown in Section 3.4, this whole process enables $\chi_{a,b,e,f}^{1,p'}(L,m)$ to be expressed in terms of various $\chi_{a',b',e,f}^{1,p'-1}(m,m')$.

Using the techniques of the previous sections, constant-sign expressions for $\chi_{a,b,c}^{1,p'}(L)$ are obtained in Section 4 by employing a succession of $\mathcal{B}$-transforms. It turns out that this may be accomplished in four different ways and these lead to four different constant-sign expressions. Two of these are derived in Sections 4.1 and 4.2. The two constant-sign expressions that result are necessarily equal, although this is by no means obvious[13]. Using a simple symmetry argument, these expressions yield the other two constant-sign expressions.

Finally, in Section 5, the relationship between the ABF and parafermion models is employed to obtain the constant-sign expressions for the ABF one-point functions, that had been conjectured by Melzer [20].

In Appendix A, we describe the $\boldsymbol{mn}$-systems that pertain to the constant-sign expressions obtained. In Appendix B, we discuss the aforementioned connection with partitions satisfying prescribed hook-difference conditions.

## 2. Combinatorics of highest weight modules

**2.1. Paths.** Let $p' \in \mathbb{N}$ with $p' \geq 2$. A path $h$ of length $L$ is a sequence $h_0, h_1, h_2, \ldots, h_L$, of integer heights such that $1 \leq h_i \leq p' - 1$ for $0 \leq i \leq L$, and such that $h_{i+1} = h_i \pm 1$ for $0 \leq i < L$. Such paths may readily be depicted on a two-dimensional $L \times (p' - 1)$ grid. The path is then the series of contiguous line

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[11]Although a similar notion, the definition of the striking sequence given in [15] differs from that used here.

[12]Our $\mathcal{B}$-transform, and the subsequent insertion of particles are all direct extensions of ideas that we learnt from [1, 7]. The 'particle moves' also appear in [1, 7] in the context of somewhat different models. They appear in the context of the ABF models in [24, 25].

[13]An analytic proof of this fact is obtained in [22].

segments passing from $(i, h_i)$ to $(i+1, h_{i+1})$ for $0 \le i < L$. Note that each of these line segments is either in the NE direction or in the SE direction. It will be useful to define the length function: $L(h) = L$.

The following is a typical path $h$. It's length is $L(h) = 11$.
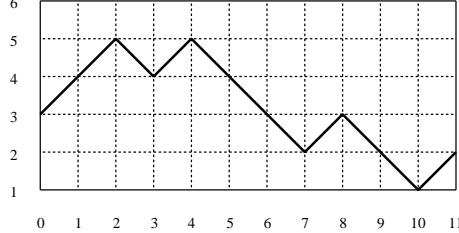


Figure 1.

If $L + a - b$ is even, define $\mathcal{P}^{p'}_{a,b}(L)$ to be the set of all paths $h$ of length $L$ with $h_0 = a$ and $h_L = b$.

In [3] and [9], a number of ways of assigning a weight to each path are described. In this paper, we are interested in only two of these. They are the cases considered in [3], and therein denoted regime III and regime II[14]. As we mentioned above, we shall refer to these as the ABF model and the parafermion model respectively.

In each case, a weight is assigned to each path $h$ only after an *extra* point $h_{L+1}$ satisfying $1 \le h_{L+1} \le p' - 1$ and $h_{L+1} = h_L \pm 1$ is specified. Then, if $1 \le a, b, c \le p' - 1$ with $c = b \pm 1$ and $L \ge 0$ is such that $L + a - b$ is even, we define $\mathcal{P}^{p'}_{a,b,c}(L)$ to be the set[15] of all paths $h$ of length L, such that $h_0 = a$, $h_L = b$ and $h_{L+1} = c$.

**2.2. ABF.** In regime III of [3] (the ABF model), each path $h$ is assigned a weight $wt^{\text{III}}(h)$ given by:

$$(1) \qquad wt^{\text{III}}(h) = \sum_{i=1}^{L} ic^{\text{III}}(h_{i-1}, h_i, h_{i+1}),$$

where, the function $c^{\text{III}}(h_{i-1}, h_i, h_{i+1})$ is defined by:

$$\begin{aligned}
c^{\text{III}}(h-1, h, h+1) &= 1/2\,; \\
c^{\text{III}}(h+1, h, h-1) &= 1/2\,; \\
c^{\text{III}}(h-1, h, h-1) &= 0\,; \\
c^{\text{III}}(h+1, h, h+1) &= 0\,.
\end{aligned}$$

Note that these four cases correspond to the four different vertex shapes. They appear as follows.

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[14]In [9], the many ways of assigning weights are indexed by $p$ with $1 \le p < p'$ and $p$ coprime to $p'$ (see also [15]). Regime III of [3] is then the case $p = p' - 1$ and regime II of [3] is the case $p = 1$.

[15]We maintain a clear distinction between $\mathcal{P}^{p'}_{a,b}(L)$ and $\mathcal{P}^{p'}_{a,b,c}(L)$. Namely, $h_{L+1}$ is defined for each element $h$ of the latter set, whereas it is not for the former set. This implies that the $L$th vertex of $h$ has a definite shape for each element of the latter, but not for the former.
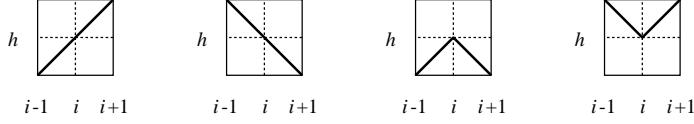
Figure 2.

They will be referred to as a *straight-up* vertex, a *straight-down* vertex, a *peak-up* vertex and a *peak-down* vertex respectively.

The generating function for paths in the ABF model is defined to be:

$$(2) \qquad X_L^{p'}(a,b,c;q) = \sum_{h \in \mathcal{P}_{a,b,c}^{p'}(L)} q^{wt^{\,\mathrm{III}}(h)}.$$

We set $X_L^{p'}(a,b,c) = X_L^{p'}(a,b,c;q)$. A bosonic (i.e. alternating sign) expression for $X_L^{p'}(a,b,c)$ is obtained in [3] (Theorem 2.3.1.). It gives

$$(3) \qquad X_L^{p'}(a,b,c) = q^{-\frac{1}{4}(a-b)(a-c)} \chi_{a,b,c}^{p'-1,p'}(L),$$

where:

$$\chi_{a,b,c}^{p'-1,p'}(L) = \sum_{\lambda=-\infty}^{\infty} q^{\lambda\left((p'-1)(\lambda p'-a)+p'r\right)} \begin{bmatrix} L \\ \frac{L+a-b}{2} - p'\lambda \end{bmatrix}_q$$

$$(4) \qquad\qquad - \sum_{\lambda=-\infty}^{\infty} q^{(\lambda p'-\lambda+r)(\lambda p'+a)} \begin{bmatrix} L \\ \frac{L+a-b}{2} - p'\lambda - a \end{bmatrix}_q,$$

where $r = (b+c-1)/2$ (i.e. $r = \min(b,c)$), and as usual, the Gaussian polynomial $\begin{bmatrix} A \\ B \end{bmatrix}_q$ is defined to be:

$$(5) \qquad \begin{bmatrix} A \\ B \end{bmatrix}_q = \frac{\prod_{i=1}^{A}(1-q^i)}{\prod_{i=1}^{B}(1-q^i)\prod_{i=1}^{A-B}(1-q^i)}$$

for $0 \le B \le A$, and $\begin{bmatrix} A \\ B \end{bmatrix}_q = 0$ otherwise.

**2.3. Parafermions.** In regime II of [3] (the parafermion model), each path $h$ is assigned a weight $wt^{\,\mathrm{II}}(h)$ given by:

$$(6) \qquad wt^{\,\mathrm{II}}(h) = \sum_{i=1}^{L} ic^{\mathrm{II}}(h_{i-1}, h_i, h_{i+1}),$$

where the function $c^{\mathrm{II}}(h_{i-1}, h_i, h_{i+1})$ is defined by:

$$\begin{aligned}
c^{\mathrm{II}}(h-1, h, h+1) &= 0\,; \\
c^{\mathrm{II}}(h+1, h, h-1) &= 0\,; \\
c^{\mathrm{II}}(h-1, h, h-1) &= 1/2\,; \\
c^{\mathrm{II}}(h+1, h, h+1) &= 1/2\,.
\end{aligned}$$

The generating function for paths in the parafermion model is defined to be:

$$(7) \qquad x_L^{p'}(a,b,c;q) = \sum_{h \in \mathcal{P}_{a,b,c}^{p'}(L)} q^{wt^{\,\mathrm{II}}(h)}.$$

Then define $x_L^{p'}(a,b,c) = x_L^{p'}(a,b,c;q)$. We immediately obtain:

LEMMA 2.1.

(8) $$x_L^{p'}(a, b, c; q) = q^{\frac{1}{4}L(L+1)} X_L^{p'}(a, b, c; q^{-1}).$$

*Proof:* For each $h \in \mathcal{P}_{a,b,c}^{p'}(L)$,

$$wt^{\,\mathrm{II}}(h) + wt^{\,\mathrm{III}}(h) = \sum_{i=1}^{L} i(c^{\mathrm{II}}(h_{i-1}, h_i, h_{i+1}) + c^{\mathrm{III}}(h_{i-1}, h_i, h_{i+1})).$$

Since for each $i$, one of $c^{\mathrm{II}}h_{i-1}, h_i, h_{i+1})$ and $c^{\mathrm{III}}(h_{i-1}, h_i, h_{i+1})$ is 1/2 and the other is 0, we get

$$wt^{\,\mathrm{II}}(h) + wt^{\,\mathrm{III}}(h) = \sum_{i=1}^{L} i/2 = L(L+1)/4.$$

The result now follows from (2) and (7). □

**2.4. An alternative prescription for weights.** In this section, we define the weight of a path in yet another way. However, as we will see, the difference between that given here and that given above is just an overall factor.

The new definition of the weight involves the path picture. Consider paths in the set $\mathcal{P}_{a,b,c}^{p'}(L)$, and define new coordinates on the grid as follows:

$$x = \frac{i - (h - a)}{2}, \qquad y = \frac{i + (h - a)}{2}.$$

Thus, the $xy$-coordinate system has its origin at the path's initial point, and is slanted at $45^o$ to the original $ih$-coordinate system. Note that at each step in the path, either $x$ or $y$ is incremented and the other is constant. In this system, the path depicted in Fig. 1 has its first few coordinates at $(0,0)$, $(0,1)$, $(0,2)$, $(1,2)$, $(1,3)$, $(2,3)$, $(3,3)$, $(4,3)$, .... Now, if the $i$th vertex has coordinates $(x, y)$, we define $c(h_{i-1}, h_i, h_{i+1})$ according to the shape of the vertex as follows:

$$\begin{aligned} c(h-1, h, h+1) &= 0; \\ c(h+1, h, h-1) &= 0; \\ c(h-1, h, h-1) &= x; \\ c(h+1, h, h+1) &= y. \end{aligned}$$

We shall refer to those vertices for which, in general, the contribution is non-zero, as *scoring* vertices. The other vertices will be termed *non-scoring*.

We now define

(9) $$\mathrm{wt}\,(h) = \sum_{i=1}^{L} c(h_{i-1}, h_i, h_{i+1}).$$

To illustrate this procedure, consider again the path $h$ depicted in Fig. 1 and take $c = 3$. The above table indicates that there are scoring vertices at $i = 2, 3, 4, 7, 8, 10$. This leads to

$$\mathrm{wt}\,(h) = 0 + 2 + 1 + 3 + 4 + 4 = 14.$$

We now define the generating function:

(10) $$\chi_{a,b,c}^{1,p'}(L; q) = \sum_{h \in \mathcal{P}_{a,b,c}^{p'}(L)} q^{\mathrm{wt}\,(h)},$$

and set $\chi_{a,b,c}^{1,p'}(L) = \chi_{a,b,c}^{1,p'}(L;q)$.

We note a symmetry of these generating functions:

LEMMA 2.2. *Let* $L \geq 0$, $1 \leq a, b < p'$ *and* $c = b \pm 1$. *Then*

$$\chi_{a,b,c}^{1,p'}(L) = \chi_{p'-a,p'-b,p'-c}^{1,p'}(L).$$

*Proof:* Let $h'$ be the path obtained from $h$ by reflecting it in a horizontal axis so that $h'_i = p' - h_i$. We immediately see that $\mathrm{wt}\,(h') = \mathrm{wt}\,(h)$. The lemma then follows from the definition (10). $\qquad\square$

LEMMA 2.3. *Let* $L \geq 0$ *and* $1 \leq a, b < p'$. *Then*

(11) $$\chi_{a,b,b\pm1}^{1,p'}(L) = q^{-\frac{1}{4}(L\pm(a-b))} x_L^{p'}(a, b, b\pm1)$$

*Proof:* Let $h \in \mathcal{P}_{a,b,c}^{p'}(L)$ and let $h$ have $N$ scoring vertices. Let the $i$ coordinates of these vertices be $i_1, i_2, \ldots, i_N$, with $1 \leq i_N < i_{N-1} < \cdots < i_1 \leq L$. Then let $(x_j, y_j)$ be the $(x,y)$-coordinates of the peak at $(i_j, h_{i_j})$.

If $c = b+1$, then there is a peak-down vertex at $i = i_1$. Then the $x$-coordinates of $(i_1, h_{i_1})$ and $(L, b)$ are equal so that $x_1 = \frac{1}{2}(L + (a-b))$. Furthermore, $y_1 = y_2$, $x_2 = x_3$, etc., so that $x_j = x_{j+1}$ for $j$ even, and $y_j = y_{j+1}$ for $j$ odd; with finally $x_N = 0$ if $N$ is even, and $y_N = 0$ if $N$ is odd. Thereupon:

$$
\begin{aligned}
i_1 + i_2 + &\cdots + i_N \\
&= (x_1 + y_1) + (x_2 + y_2) + \cdots + (x_N + y_N) \\
&= \frac{1}{2}(L + (a-b)) + 2y_1 + 2x_2 + 2y_3 + \cdots + \begin{cases} 2x_N & \text{if } N \text{ is even,} \\ 2y_N & \text{if } N \text{ is odd} \end{cases} \\
&= \frac{1}{2}(L + (a-b)) + 2\,\mathrm{wt}\,(h).
\end{aligned}
$$

On the other hand, if $c = b - 1$ whereupon $y_1 = \frac{1}{2}(L - (a-b))$, a similar argument (in fact, by just exchanging the roles of $x$ and $y$ in the above) leads to:

$$i_1 + i_2 + \cdots + i_N = \frac{1}{2}(L - (a-b)) + 2\,\mathrm{wt}\,(h).$$

Combining these two results thus yields:

$$\mathrm{wt}^{\,\mathrm{II}}(h) = \frac{1}{4}(L \pm (a-b)) + \mathrm{wt}\,(h),$$

when $b = c \pm 1$. The lemma then follows immediately from the definitions (7) and (10). $\qquad\square$

The following result now provides the relationship between the renormalised ABF and parafermion generating functions.

LEMMA 2.4. *Let* $L \geq 0$, $1 \leq a, b < p'$ *and* $c = b \pm 1$. *Then*

$$\chi_{a,b,c}^{1,p'}(L;q) = q^{\frac{1}{4}(L^2-(a-b)^2)} \chi_{a,b,c}^{p'-1,p'}(L;q^{-1}).$$

*Proof:* This result follows from combining the definition (3) with Lemmas 2.1 and 2.3. $\qquad\square$

**2.5. Striking sequence of a path.** Scanning from left to right, one can think of each $h \in \mathcal{P}^{p'}_{a,b}(L)$ as a sequence of straight lines, alternating in direction between NE and SE. Let the lengths of these lines be $w_1, w_2, w_3, \ldots, w_l$, for some $l$, so that $w_1 + w_2 + \cdots + w_l = L(h)$. In what follows, we permit $w_1 = 0$ and $w_l = 0$, but restrict $w_i > 0$ for $1 < i < l$.

As will become clear shortly, we need to augment the definition of a path as follows: for each path, we fix $e, f \in \{0, 1\}$ (arbitrarily for now), and require $w_1$ to be the number of SE (resp. NE) segments at the beginning of the path if $e = 0$ (resp. $e = 1$), and $w_l$ to be the number of NE (resp. SE) segments at the end of the path if $f = 0$ (resp. $f = 1$). Notice that there are 4 possible augmentations of each path. This definition implies that $l \equiv e + f \pmod{2}$. The striking sequence of the 'augmented' path, $h$, is then defined to be the symbol:

$$(12) \qquad (w_1, w_2, w_3, \ldots, w_l)^{(e,f)}.$$

We now define $m^{(e,f)}(h) = L - l + 2$, where $l$ is the number of elements in the striking sequence above. Note that $l \equiv e + f \pmod{2}$ implies that $m^{(e,f)}(h) \equiv L + e + f \pmod{2}$. For a given path $h$, we see that no two values of $m^{(0,0)}(h)$, $m^{(0,1)}(h)$, $m^{(1,0)}(h)$ and $m^{(1,1)}(h)$ are guaranteed equal.

For example, the path $h$ shown in Fig. 1 has the four possible striking sequences: $(0, 2, 1, 1, 3, 1, 2, 1)^{(0,0)}$, $(2, 1, 1, 3, 1, 2, 1)^{(1,0)}$, $(0, 2, 1, 1, 3, 1, 2, 1, 0)^{(0,1)}$ and $(2, 1, 1, 3, 1, 2, 1, 0)^{(1,1)}$. Thence, we obtain $m^{(0,0)}(h) = 5$, $m^{(1,0)}(h) = 6$, $m^{(0,1)}(h) = 4$ and $m^{(1,1)}(h) = 5$.

LEMMA 2.5. *Let $h \in \mathcal{P}^{p'}_{a,b}(L)$ and let $e, f \in \{0, 1\}$.*

- $m^{(1,f)}(h) = m^{(0,f)}(h) + 1$, *for $a = 1$;*
- $m^{(1,f)}(h) = m^{(0,f)}(h) - 1$, *for $a = p' - 1$;*
- $m^{(e,1)}(h) = m^{(e,0)}(h) + 1$, *for $b = 1$;*
- $m^{(e,1)}(h) = m^{(e,0)}(h) - 1$, *for $b = p' - 1$.*

*Proof:* If $a = 1$, then the first segment of the path is certainly in the NE direction. Thus, with $w_1 > 0$, the path $h$ has striking sequences $(w_1, w_2, \ldots, w_l)^{(1,f)}$ and $(0, w_1, w_2, \ldots, w_l)^{(0,f)}$. Then $m^{(1,f)} = L - l + 2$ and $m^{(0,f)} = L - (l + 1) + 2$, whereupon the first result follows immediately. The other three results follow in an analogous way. $\square$

The purpose of assigning $e$ and $f$ to a path $h$ of length $L$, is to enable the 0th and $L$th vertices to be each designated as scoring or non-scoring. In fact, if these vertices are included, $m^{(e,f)}(h)$ gives the total number of non-scoring vertices in $h$. We see that prescribing $e$ and $f$ is equivalent to appending two extra segments to the path, a *pre-segment* that ends at the 0th vertex, and a *post-segment* that starts at the $L$th vertex. Setting $e = 0$ (resp. $e = 1$) is equivalent to having a pre-segment that points SE (resp. NE). Setting $f = 0$ (resp. $f = 1$) is equivalent to having a post-segment that points NE (resp. SE).

If we refer to these additional segments as the 0th and $(L + 1)$th segments respectively, then the above definition of the striking sequence implies that $w_1$ counts the number of segments at the beginning of the path in the same direction as (but not including) the 0th and $w_l$ counts the number of segments at the end of the path in the same direction as (but not including) the $(L + 1)$th. Note that specifying $f$ is equivalent to specifying the extra point $h_{L+1}$.

We now define a weight for each $h \in \mathcal{P}_{a,b}^{p'}(L)$ that depends on $e$ and $f$, and then show that this weight is equal to the weight of the corresponding path in $\mathcal{P}_{a,b,c}^{p'}(L)$ with $c$ appropriately defined.

DEFINITION 2.6. Let $h \in \mathcal{P}_{a,b}^{p'}(L)$, let $e, f \in \{0, 1\}$ and let the path $h$ have the striking sequence $(w_1, w_2, w_3, \ldots, w_l)^{(e,f)}$. Then define

$$\mathrm{wt}^{(e,f)}(h) = \sum_{i=2}^{l-1}(w_{i-1} + w_{i-3} + \cdots + w_{1+i \bmod 2}).$$

We now show that this definition essentially provides the weight of the corresponding path for which the appropriate extra point $h_{L+1}$ is defined.

LEMMA 2.7. Let $h \in \mathcal{P}_{a,b}^{p'}(L)$ and let $e, f \in \{0, 1\}$. If $f = 0$ then let $c = b + 1$ and if $f = 1$ then let $c = b - 1$. Then let $h' \in \mathcal{P}_{a,b,c}^{p'}(L)$ be such that $h_i' = h_i$ for $0 \leq i \leq L$. Then $\mathrm{wt}(h') = \mathrm{wt}^{(e,f)}(h)$.

*Proof:* Let the path $h$ have the striking sequence $(w_1, w_2, w_3, \ldots, w_l)^{(e,f)}$, where $w_1 \geq 0$, $w_l \geq 0$ and $w_i > 0$ for $1 < i < l$.

Except for $i = l$ and possibly $i = 1$, there is a scoring vertex at the end of the $i$th line (which has length $w_i$) of $h'$. First assume that the first $w_1$ segments of $h$ are in the NE direction. Then, for $i$ odd, the $i$th line is in the NE direction and its $x$-coordinate is $w_2 + w_4 + \cdots + w_{i-1}$. By the prescription of the previous section, this line thus contributes $(w_2 + w_4 + \cdots + w_{i-1})$ to the weight $\mathrm{wt}(h')$ of $h'$. Similarly, for $i$ even, the $i$th line is in the SE direction and contributes $(w_1 + w_3 + \cdots + w_{i-1})$ to $\mathrm{wt}(h')$. This proves the lemma if the first $w_1$ segments are in the NE direction. The reasoning is almost identical for the other case.                                    $\square$

LEMMA 2.8. Let $h \in \mathcal{P}_{a,b}^{p'}(L)$ and $e, f \in \{0, 1\}$. Then $\mathrm{wt}^{(0,f)}(h) = \mathrm{wt}^{(1,f)}(h)$. Furthermore,

- $\mathrm{wt}^{(e,0)}(h) = \mathrm{wt}^{(e,1)}(h) + \frac{1}{2}(L - a + 1)$, for $b = 1$;
- $\mathrm{wt}^{(e,0)}(h) = \mathrm{wt}^{(e,1)}(h) - \frac{1}{2}(L - p' + a)$, for $b = p' - 1$.

*Proof:* For either $e = 0$ or $e = 1$, $h$ has striking sequence $(w_1, w_2, \ldots, w_l)^{(e,f)}$ with $w_1 > 0$. Then $h$ also has striking sequence $(0, w_1, w_2, \ldots, w_l)^{(1-e,f)}$. Definition 2.6 then gives $\mathrm{wt}^{(e,f)}(h) = \mathrm{wt}^{(1-e,f)}(h)$, thereby proving the first part.

If $b = 1$, then the $L$th segment of $h$ is necessarily in the SE direction. Therefore $h$ has striking sequence $(w_1, w_2, \ldots, w_l)^{(e,1)}$ with $w_l > 0$. It then also has striking sequence $(w_1, w_2, \ldots, w_l, 0)^{(e,0)}$. Therefore, $\mathrm{wt}^{(e,0)}(h) - \mathrm{wt}^{(e,1)}(h) = w_{l-1} + w_{l-3} + w_{l-5} + \cdots$. Since $w_1 + w_2 + \cdots + w_l = L$ and $(w_l + w_{l-2} + \cdots) - (w_{l-1} + w_{l-3} + \cdots) = a - 1$, it follows that $\mathrm{wt}^{(e,0)}(h) - \mathrm{wt}^{(e,1)}(h) = \frac{1}{2}(L - a + 1)$.

A similar argument for the $b = p' - 1$ case, shows that $h$ has striking sequences $(w_1, w_2, \ldots, w_l)^{(e,0)}$ and $(w_1, w_2, \ldots, w_l, 0)^{(e,1)}$, where $w_l > 0$. Then $\mathrm{wt}^{(e,1)}(h) - \mathrm{wt}^{(e,0)}(h) = w_{l-1} + w_{l-3} + w_{l-5} + \cdots$. Now $(w_l + w_{l-2} + \cdots) - (w_{l-1} + w_{l-3} + \cdots) = p' - a + 1$, whereupon $\mathrm{wt}^{(e,1)}(h) - \mathrm{wt}^{(e,0)}(h) = \frac{1}{2}(L - p' + a)$.                $\square$

**2.6. Restricted generating functions.** We now define the set of paths $\mathcal{P}^{p'}_{a,b,e,f}(L,m)$ to be the subset of $\mathcal{P}^{p'}_{a,b}(L)$, comprising those paths $h$ for which $m^{(e,f)}(h) = m$. Let $\chi^{1,p'}_{a,b,e,f}(L,m)$ be the generating function for all such paths:

$$\chi^{1,p'}_{a,b,e,f}(L,m) = \sum_{h \in \mathcal{P}^{p'}_{a,b,e,f}(L,m)} q^{\text{wt}\,(e,f)(h)}.$$

Note that $\chi^{1,p'}_{a,b,e,f}(L,m) = 0$ unless $m \equiv L + e + f \,(\text{mod}\,2)$.

LEMMA 2.9. *Let $1 \leq a,b < p'$ and $e,f \in \{0,1\}$. Then:*

- $\chi^{1,p'}_{1,b,1,f}(L,m) = \chi^{1,p'}_{1,b,0,f}(L,m-1)$;
- $\chi^{1,p'}_{p'-1,b,1,f}(L,m) = \chi^{1,p'}_{p'-1,b,0,f}(L,m+1)$;
- $\chi^{1,p'}_{a,1,e,0}(L,m) = q^{\frac{1}{2}(L-a+1)}\chi^{1,p'}_{a,1,e,1}(L,m+1)$;
- $\chi^{1,p'}_{a,p'-1,e,0}(L,m) = q^{-\frac{1}{2}(L-p'+a)}\chi^{1,p'}_{a,p'-1,e,1}(L,m-1)$.

*Proof:* If $h \in \mathcal{P}^{p'}_{1,b,1,f}(L,m)$ then $h \in \mathcal{P}^{p'}_{1,b}(L)$ and $m = m^{(1,f)}(h)$. Then by Lemma 2.5, $m^{(0,f)}(h) = m-1$, so that $\mathcal{P}^{p'}_{1,b,0,f}(L,m-1) \subset \mathcal{P}^{p'}_{1,b,1,f}(L,m)$. On reversing the argument, the direction of the inclusion here is changed, whereupon $\mathcal{P}^{p'}_{1,b,0,f}(L,m-1) = \mathcal{P}^{p'}_{1,b,1,f}(L,m)$. By Lemma 2.8, wt $^{(0,f)}(h) = $ wt $^{(1,f)}(h)$ for all $h \in \mathcal{P}^{p'}_{1,b}(L)$. Thereupon, the first statement of the lemma is proved. The second statement follows in a similar way.

For the third statement, we also obtain $\mathcal{P}^{p'}_{a,1,e,0}(L,m-1) = \mathcal{P}^{p'}_{a,1,e,1}(L,m)$ in a similar way. However, Lemma 2.8 gives wt $^{(e,0)}(h) = $ wt $^{(e,1)}(h) + \frac{1}{2}(L-a+1)$ for all $h \in \mathcal{P}^{p'}_{a,1}(L)$. Thereupon, the third statement follows. The fourth statement follows similarly.                                                                              □

The following result will act as a seed to generate further expressions.

LEMMA 2.10. *Let $1 \leq a,b < p'$ with $p' \geq 2$. In addition, let $e,f \in \{0,1\}$. Then*

$$\chi^{1,p'}_{a,b,e,f}(0,m) = \delta_{a,b}\delta_{|e-f|,m}.$$

*If $L \geq 0$, then*

$$\chi^{1,2}_{1,1,e,f}(L,m) = \delta_{L,0}\delta_{|e-f|,m}.$$

*Proof:* Clearly $\mathcal{P}^{p'}_{a,b}(0) = \emptyset$ if $a \neq b$. Otherwise, it contains a single element. Let $h$ designate this path. In the case $e = f$, $h$ has striking sequence $(0,0)^{(e,f)}$ whereupon $m^{(e,f)}(h) = 0$ immediately from the definition. We also obtain wt $^{(e,f)}(h) = 1$ whereupon the result follows for $e = f$. In the case $e \neq f$, $h'$ has striking sequence $(0)^{(e,f)}$. Then $m^{(e,f)}(h) = 1$ and wt $^{(e,f)}(h) = 1$ whence the required result also follows for $e \neq f$.

For the second expression, it is clear that $\mathcal{P}^2_{a,b}(L) = \emptyset$ unless $L = 0$. The result then follows from the first part.                                                                              □

LEMMA 2.11. *Let $1 \leq a,b < p'$ and $c = b \pm 1$. Then if $c = b+1$ let $f = 0$, and if $c = b-1$ let $f = 1$. Then for each $e \in \{0,1\}$,*

$$\chi^{1,p'}_{a,b,c}(L) = \sum \chi^{1,p'}_{a,b,e,f}(L,m)$$

*where the sum is over all m for which $m \equiv L + e + f \,(mod\,2)$.*

*Proof:* For each $h' \in \mathcal{P}^{p'}_{a,b,c}(L)$, there is a corresponding path $h \in \mathcal{P}^{p'}_{a,b}(L)$ for which $h'_i = h_i$ for $0 \le i \le L$, and vice-versa. Moreover, $h \in \mathcal{P}^{p'}_{a,b,e,f}(L,m)$ for one and only one value of $m$ which is given by $m = m^{(e,f)}(h)$, whereupon $m \equiv L + e + f \,(\mathrm{mod}\,2)$. Then $\mathcal{P}^{p'}_{a,b,c}(L) = \bigcup_{m \equiv L+e+f \,(\mathrm{mod}2)} \mathcal{P}^{p'}_{a,b,e,f}(L,m)$. Since, by Lemma 2.7, wt $(h') =$ wt $^{(e,f)}(h)$, the current lemma follows. □

## 3. Path transformations

**3.1. $\mathcal{B}$-transforms.** In this section, we define a method of transforming a path in $\mathcal{P}^{p'}_{a,b}(L)$ to yield one in $\mathcal{P}^{p'+1}_{a',b'}(L')$ for certain $a'$, $b'$ and $L'$. This transform will be referred to as a $\mathcal{B}$-transform[16].

The action of the $\mathcal{B}$-transform is most easily described using the striking sequences. For $e, f \in \{0,1\}$, the action of the $\mathcal{B}$-transform on the path $h$ described by the striking sequence $(w_1, w_2, w_3, \ldots, w_l)^{(e,f)}$ is to yield the path $\hat{h}$ with $\hat{h}_0 = h_0 + e$ and striking sequence $(w_1, w_2 + 1, w_3 + 1, \ldots, w_{l-1} + 1, w_l)^{(e,f)}$. Note that the $\mathcal{B}$-transform action is dependent on the values of $e$ and $f$ that appear in the striking sequence of the path.

For example, if $e = f = 0$ then the action of the $\mathcal{B}$-transform on the path given in Fig. 1 results in the path:
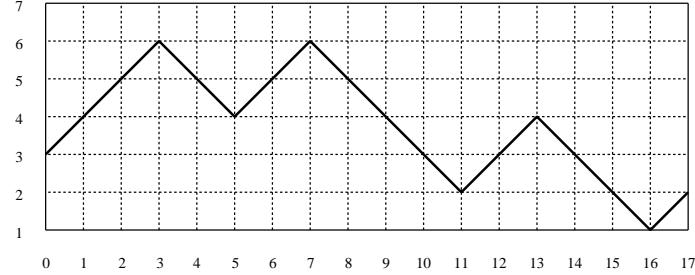


and if $e = 1$ and $f = 0$ then the action of the $\mathcal{B}$-transform on the path given in Fig. 1 results in the path:
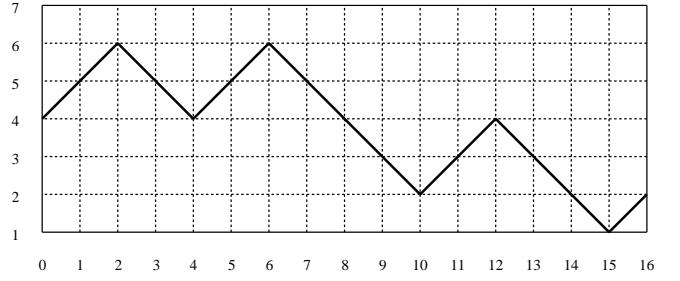


Note that the path obtained from the action of the $\mathcal{B}$-transform is such that there are no two consecutive scoring vertices.

---

[16]It may be seen that when $e = f = 0$, the $\mathcal{B}$-transform described here is a generalisation of the $\mathcal{B}$-transform described in [**15**] as it acts upon paths in $\mathcal{P}^{1,p'}_{1,1}(L)$.

Special care must be taken when dealing with paths of length 0 when $e \neq f$. The striking sequence of such a path is $(0)$. We choose to leave the action of the $\mathcal{B}$-transform on such a path undefined. Then Lemmas 3.1, 3.2, 3.3, 3.4, 3.5 and 3.6 will not apply for such paths. However, they appear as a special case in the proof of Lemma 3.7.

LEMMA 3.1. *Let $h \in \mathcal{P}_{a,b}^{p'}(L)$ and, for $e, f \in \{0,1\}$, let $\hat{h}$ be the path obtained from the action of the $\mathcal{B}$-transform on $h$. Then $\hat{h} \in \mathcal{P}_{a+e,b+f}^{p'+1}(\hat{L})$, $m^{(e,f)}(\hat{h}) = L$ and $L(\hat{h}) = \hat{L} = 2L - m^{(e,f)}(h)$.*

*Proof:* From the definition of the $\mathcal{B}$-transform, we immediately obtain $\hat{L} = L + l - 2$, whereupon $m^{(e,f)}(h) = L - l + 2$ implies that $\hat{L} = 2L - m^{(e,f)}(h)$. Additionally $m^{(e,f)}(\hat{h}) = \hat{L} - l + 2 = L$. Now set $\hat{a} = \hat{h}_0$ and $\hat{b} = \hat{h}_{\hat{L}}$. The definition of the $\mathcal{B}$-transform immediately gives $\hat{a} = a + e$. In terms of the striking sequence $(w_1, w_2, \ldots, w_l)^{(e,f)}$ of $h$, we have $a - b = (-1)^e((w_1 + w_3 + \cdots) - (w_2 + w_4 + \cdots))$. Then if $l$ is even, $\hat{a} - \hat{b} = a - b$, and if $l$ is odd $\hat{a} - \hat{b} = a - b - (-1)^e$. Using $l \equiv e + f \pmod 2$ gives $\hat{a} - \hat{b} = a - b + e - f$ in both cases and hence $\hat{b} = b + f$ as required.                                                  $\square$

LEMMA 3.2. *Let $h \in \mathcal{P}_{a,b}^{p'}(L)$ and, for $e, f \in \{0,1\}$, let $\hat{h}$ be the path obtained from the action of the $\mathcal{B}$-transform on $h$. Then,*

$$\mathrm{wt}^{(e,f)}(\hat{h}) = \mathrm{wt}^{(e,f)}(h) + \frac{1}{4}((\hat{L} - \hat{m})^2 - \delta_{e+f,1}),$$

*where $\hat{L} = L(\hat{h})$ and $\hat{m} = m^{(e,f)}(\hat{h})$.*

*Proof:* Let the striking sequence of $h$ be $(w_1, w_2, \ldots, w_{l-1}, w_l)^{(e,f)}$ whereupon that of $\hat{h}$ is $(w_1, w_2 + 1, \ldots, w_{l-1} + 1, w_l)^{(e,f)}$. Definition 2.7 then gives:

$$\mathrm{wt}^{(e,f)}(\hat{h}) - \mathrm{wt}^{(e,f)}(h) = 0 + 1 + 1 + 2 + 2 + 3 + \cdots + \lfloor (l-3)/2 \rfloor + \lfloor (l-2)/2 \rfloor.$$

This sum is $\frac{1}{4}(l-2)^2$ if $l$ is even and $\frac{1}{4}((l-2)^2 - 1)$ if $l$ is odd. The result then follows because $L(\hat{h}) - m^{(e,f)}(\hat{h}) = l - 2$ and $l \equiv e + f \pmod 2$.                 $\square$

### 3.2. Inserting particles.

Given a path $h^{(0)}$ of length $L$ and striking sequence $(w_1, w_2, \ldots, w_l)^{(e,f)}$, we may extend $h^{(0)}$ by a process we refer to as *inserting particles*. If $a = 1$, we restrict this process to the $e = 0$ case, and if $a = p' - 1$, we restrict to the $e = 1$ case. The effect of inserting one particle is to produce a path $h^{(1)}$ with the same starting point and striking sequence $(0, 1, w_1 + 1, w_2, \ldots, w_l)^{(e,f)}$. Thus the new path has length $L + 2$. Notice that the way that the path is extended depends on $e$. By iterating the process, we may insert $k$ particles into $h^{(0)}$ to obtain a path $h^{(k)}$ of length $L + 2k$.

LEMMA 3.3. *Let $h \in \mathcal{P}_{a,b}^{p'}(L)$ and, for $e, f \in \{0,1\}$, let $h^{(0)}$ be the path obtained from the action of the $\mathcal{B}$-transform on $h$, and obtain $h^{(k)}$ from $h^{(0)}$ by inserting $k$ particles. If $m = m^{(e,f)}(h)$, $m' = m^{(e,f)}(h^{(k)})$ and $L' = L(h^{(k)})$, then $m' = L$,*

$$L' + m = 2m' + 2k,$$

*and*

$$\mathrm{wt}^{(e,f)}(h^{(k)}) = \mathrm{wt}^{(e,f)}(h) + \frac{1}{4}\left((L' - m')^2 - \delta_{e+f,1}\right).$$

*Proof:* By Lemma 3.1, $L(h^{(0)}) = 2L - m$ and $m^{(e,f)}(h^{(0)}) = L$. Inserting $k$ particles then gives $L' = L(h^{(0)}) + 2k = 2L - m + 2k$ and $m' = m^{(e,f)}(h^{(k)}) = m^{(e,f)}(h^{(0)}) = L$.

To obtain the final result, let $h^{(0)}$ have striking sequence $(w_1, w_2, \ldots, w_l)^{(e,f)}$, whereupon that of $h^{(1)}$ is $(0, 1, w_1 + 1, w_2, \ldots, w_l)^{(e,f)}$. Then, $m^{(e,f)}(h^{(1)}) = m^{(e,f)}(h^{(0)})$ and Definition 2.7 gives $\mathrm{wt}^{(e,f)}(h^{(1)}) = \mathrm{wt}^{(e,f)}(h^{(0)}) + l - 1$. Repeated application then yields $m^{(e,f)}(h^{(k)}) = m^{(e,f)}(h^{(0)})$ and

$$
\begin{aligned}
\mathrm{wt}^{(e,f)}(h^{(k)}) &= \mathrm{wt}^{(e,f)}(h^{(0)}) + k(l-1) + k(k-1) \\
&= \mathrm{wt}^{(e,f)}(h^{(0)}) + k(l-2) + k^2 \\
&= \mathrm{wt}^{(e,f)}(h^{(0)}) + k\left(L(h^{(0)}) - m^{(e,f)}(h^{(0)})\right) + k^2 \\
&= \mathrm{wt}^{(e,f)}(h) + \frac{1}{4}\left(\left(L(h^{(0)}) - m^{(e,f)}(h^{(0)})\right)^2 - \delta_{e+f,1}\right) \\
&\qquad\qquad + k\left(L(h^{(0)}) - m^{(e,f)}(h^{(0)})\right) + k^2,
\end{aligned}
$$

where the final equality follows from Lemma 3.2. The required expression now results because, from above, $L(h^{(0)}) = L' - 2k$ and $m^{(e,f)}(h^{(0)}) = m'$.  □

**3.3. Moving particles.** In this section, we specify two types of local deformation of a path. These deformations will be known as *moves*. In each case, a particular sequence of four segments of a path is changed to a different sequence, the remainder of the path being unchanged. The moves are as follows — the path portion to the left of the arrow is changed to that on the right:



Move. 1.



Move. 2.
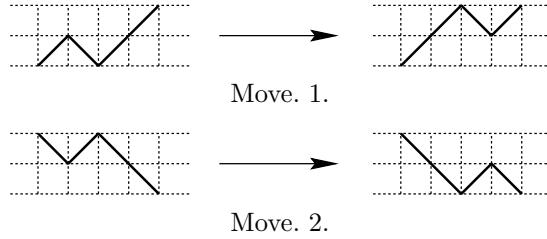
Note that the two moves are inversions of one another.

LEMMA 3.4. *Let h be a path for which four consecutive segments are as in one of diagrams on the left above. Let $\hat{h}$ be that obtained from h by changing those segments according to the move. Then, for $e, f \in \{0, 1\}$,*

$$\mathrm{wt}^{(e,f)}(\hat{h}) = \mathrm{wt}^{(e,f)}(h) + 1.$$

*Additionally, $m^{(e,f)}(\hat{h}) = m^{(e,f)}(h)$ and $L(\hat{h}) = L(h)$.*

*Proof:* For each case, take the $xy$-coordinate of the leftmost point of this portion of a path to be $(x_0, y_0)$. Now consider the contribution to the weight of the three vertices in question before and after the move In both cases, the contribution is $x_0 + y_0 + 1$ before the move and $x_0 + y_0 + 2$ afterwards. Thus the first result holds. The final result follows immediately from the definitions.  □

Now observe that for each of the moves specified above, the sequence of path segments before the move consists of an adjacent pair of scoring vertices followed

by a non-scoring vertex. The specified move then consists of replacing such a combination with a non-scoring vertex followed by two scoring vertices. It is useful to interpret this as the pair of adjacent scoring vertices having moved by one step. In fact, it is useful to refer to a pair of adjacent scoring vertices as a particle. Thus both of the moves consists of a particle moving rightwards by one step.

In addition to the moves described above, and depending on the values of $e$ and $f$, we permit certain deformations of a path close to its left and right extremities. They are as follows.

If $e = 1$ :

If $e = 0$ :

If $f = 0$ :

If $f = 1$ :
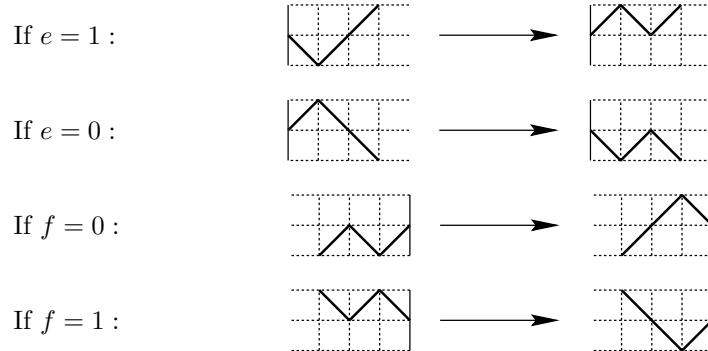
In fact, the above four moves may be considered as instances of the two moves described beforehand, if we append two extra segments to the path as described in the paragraph following Lemma 2.5.

LEMMA 3.5. *Let h be a path for which three consecutive segments occupy three extreme positions and are as in one of the diagrams above, on the left. Let $\hat{h}$ be that obtained from h by changing those segments according to the move. Then, for $e, f \in \{0, 1\}$,*

$$\mathrm{wt}^{(e,f)}(\hat{h}) = \mathrm{wt}^{(e,f)}(h) + 1.$$

*Additionally, $m^{(e,f)}(\hat{h}) = m^{(e,f)}(h)$ and $L(\hat{h}) = L(h)$.*

*Proof:* As for Lemma 3.4. □

With the interpretation of the extremity moves in terms of additional segments to the left and right of the original path, we may also designate the 0th and $L$th vertices as scoring or non-scoring. We then see that the four extremity moves also involve the replacing of a pair of scoring vertices that are followed by a non-scoring vertex, by a non-scoring vertex followed by a pair of scoring vertices.

**3.4. Waves of particles.** Since each of the moves described above involves a pair of scoring vertices moving rightwards by one step, we see that a succession of such moves is possible until the pair is followed by another scoring vertex. If this itself is followed by yet another scoring vertex, we forbid further movement. However, if it is followed by a non-scoring vertex, further movement is allowed after considering the latter two of the three consecutive scoring vertices to be the particle (instead of the first two).

Now consider a path $h^{(k)}$ that results from inserting $k$ particles into $h^{(0)}$ that itself results from the action of the $\mathcal{B}$-transform. We now show that the paths that result from moving these particles in all possible ways, are indexed by partitions (a definition of a partition may be found in Appendix B).

LEMMA 3.6. *There is a bijection between the paths obtained by moving the particles in $h^{(k)}$ and the partitions $\lambda$ with at most $k$ parts that have no part larger than $m = m^{(e,f)}(h^{(k)})$. This bijection is such that if $h$ is the bijective image of a particular $\lambda$ then*

$$\mathrm{wt}\,(h) = \mathrm{wt}\,(h^{(k)}) + \mathrm{wt}\,(\lambda),$$

*where* $\mathrm{wt}\,(\lambda) = \lambda_1 + \lambda_2 + \cdots + \lambda_k$.

*Proof:* If $h^{(0)}$ has striking sequence $(w_1, w_2, \ldots, w_l)^{(e,f)}$ then there are $l - 1$ scoring vertices between the 0th and $L$th vertex inclusive — one at the end of the $i$th line which has length $w_i$ for $1 \leq i < l$. Since there are altogether $L + 1$ vertices, the number of non-scoring vertices is $L + 1 - (l - 1) = L - l + 2 = m^{(e,f)}(h^{(0)})$.

Since each particle moves by traversing a non-scoring vertex, and there are $m$ of these to the right of the rightmost particle in $h^{(k)}$, and there are no consecutive scoring vertices to its right, this particle can make $\lambda_1$ moves to the right, with $0 \leq \lambda_1 \leq m$. Similarly, the next rightmost particle can make $\lambda_2$ moves to the right with $0 \leq \lambda_2 \leq \lambda_1$. Here, the upper restriction arises because the two scoring vertices would then be adjacent to those of the first particle. Continuing in this way, we obtain that all possible final positions of the particles are indexed by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $m \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$, that is, by partitions of at most $k$ parts with no part exceeding $m$. Moreover, since by Lemmas 3.4 and 3.5 the weight increases by one for each move, the weight increase after the sequence of moves specified by a particular $\lambda$ is equal to $\mathrm{wt}\,(\lambda)$. $\qquad\square$

We now combine the above results on the $\mathcal{B}$-transform, and the inserting and moving of particles to obtain an expression which enables the restricted generating function $\chi_{a,b,e,f}^{1,p'}(L', m')$ to be obtained from those of a 'simpler' model.

LEMMA 3.7. *Let $e, f \in \{0, 1\}$. Then, if $p' > 2$, and $1 \leq a - e, b - f < p' - 1$,*

$$\chi_{a,b,e,f}^{1,p'}(L', m') = \sum_{m \equiv L' \,(\mathrm{mod}\,2)} q^{\frac{1}{4}\left((L'-m')^2 - \delta_{e+f,1}\right)} \begin{bmatrix} \frac{1}{2}(L' + m) \\ m' \end{bmatrix}_q \chi_{a-e,b-f,e,f}^{1,p'-1}(m', m).$$

*Proof:* For the moment, exclude the case where $m' = 0$ and $L'$ is odd. Consider a path $h$ that contributes to $\chi_{a-e,b-f,e,f}^{1,p'-1}(m', m)$ so that $L(h) = m'$ and $m^{(e,f)}(h) = m$. Let $h^{(0)}$ result from the action of the $\mathcal{B}$-transform on $h$. By Lemma 3.3, inserting $k$ particles into $h^{(0)}$ results in a path $h^{(k)}$ of length $L'$ if and only if $2k = L' - 2m' + m$. (In particular, $m \equiv L' \,(\mathrm{mod}\,2)$.) The generating function for all paths obtained by moving the particles in $h^{(k)}$ is then given by

$$q^{\frac{1}{4}\left((L'-m')^2 - \delta_{e+f,1}\right)} \begin{bmatrix} k + m' \\ m' \end{bmatrix}_q q^{\mathrm{wt}\,(h)},$$

on using the last expression of Lemma 3.3, and using Lemma 3.6 after noting that the Gaussian term $\begin{bmatrix} k+m' \\ m' \end{bmatrix}$ is the generating function for all partitions $\lambda$ with $k$ non-negative parts, none exceeding $m'$. Summing this expression over all $h \in \mathcal{P}_{a-e,b-f,e,f}^{p'-1}(m', m)$ and then all $m \equiv L' \,(\mathrm{mod}\,2)$, results in the expression on the right side of that in the premise.

We now show that every path $h' \in \mathcal{P}_{a,b,e,f}^{p'}(L', m')$ arises through the above procedure and moreover, arises in a unique way. So let $h' \in \mathcal{P}_{a,b,e,f}^{p'}(L', m')$. Locate

the leftmost pair of consecutive scoring vertices in $h'$, and move them leftward by reversing the particle moves, until they occupy the 0th and 1st positions. Now ignoring these two vertices, do the same with the next leftmost pair of consecutive scoring vertices, moving them leftward until they occupy the 2nd and 3rd positions. Continue in this way until all consecutive scoring vertices occupy the leftmost positions of the path. Say there are $\kappa$ of them, and let $k = \lfloor \kappa/2 \rfloor$. Removing the first $2k$ segments then produces a path $h^{(0)}$ which, by construction, has no pair of consecutive scoring vertices. Thus, its striking sequence will be of the form $(w_1, w_2, \dots, w_{l-1}, w_l)^{(e,f)}$, with $w_i \geq 2$ for $1 < i < l$. Thus $h^{(0)}$ arises from the action of the $\mathcal{B}$-transform on the path $h$ that has striking sequence $(w_1, w_2 - 1, \dots, w_{l-1} - 1, w_l)^{(e,f)}$, and $h_0 = h'_0 - e$. Since $h$ and $k$ are thereby determined uniquely, the lemma is proved except when $m' = 0$ and $L'$ odd[17].

In this exceptional case, there is only one path $h'$ in $\mathcal{P}^{p'}_{a,b,e,f}(L', m')$ since each of its $L'+1$ vertices are scoring, and thus $h'$ has scoring sequence $(1, 1, 1, \dots, 1)^{(e,f)}$. Lemma 2.7 gives, via Definition 2.6, $\mathrm{wt}\,(h') = \frac{1}{4}(L'^2 - 1)$. The form of the striking sequence also ensures that $a - b = \pm 1$, and moreover, $a - e = b - f$, so that $|e - f| = 1$. In the sum on the right side, for other than $m = 1$, we have $\chi^{1,p'-1}_{a-e,b-f,e,f}(0, m) = 0$. However, by Lemma 2.10, $\chi^{1,p'-1}_{a-e,b-f,e,f}(0, 1) = 1$. The proof is then complete. $\square$

## 4. Parafermion generating functions

In the previous section, it was indicated that all paths in $\mathcal{P}^{p'}_{a,b}(L)$ can be obtained by applying a sequence of $\mathcal{B}$-transforms to $\mathcal{P}^2_{1,1}(L)$. We do this by repeated application of Lemma 3.7. In fact, this approach leads to four different concise constant-sign expressions for the path generating functions. We derive two of these in Sections 4.1 and 4.2.

These two expressions are equal. Combinatorially, this follows from the fact that both count the same set of objects. They look different, because we interpret these objects differently. An analytic proof of their equality can be found in [22].

The two further constant-sign expressions result from applying the symmetry obtained in Lemma 2.2. Although we do not show this, these expressions may also be obtained by using the $\mathcal{B}$-transforms in a way similar to which the first two were derived.

The expressions that we obtain involve the Cartan matrix $\boldsymbol{C} = \boldsymbol{C}^{(t)}$ of the finite dimensional Lie algebra $A_t$, i.e. $\boldsymbol{C}^{(t)}$ is the $t \times t$ tri-diagonal matrix with entries $\boldsymbol{C}_{ij}$ for $0 \leq i, j \leq t - 1$ where, when the indices are in this range,

$$(13) \qquad \boldsymbol{C}_{j,j-1} = -1, \quad \boldsymbol{C}_{j,j} = 2, \quad \boldsymbol{C}_{j,j+1} = -1, \quad \text{for } j = 0, 1, \dots, t - 1.$$

The expressions also involve $(p' - 2)$-dimensional vectors $\boldsymbol{u}_{a,b}$ which depend on $a$ and $b$. Define $\boldsymbol{u}_{a,b} = (u_1, u_2, \dots, u_{p'-2})$ as follows:

$$(14) \qquad u_i = \delta_{i,a-1} + \delta_{i,b-1},$$

for $1 \leq i \leq p' - 2$.

---

[17]The above proof fails in this case because, the only non-zero term on the right side has $m' = 0$ and $m = 1$, and the $\mathcal{B}$-transform is not defined on such paths. Moreover, in the second paragraph, we would obtain $\kappa = L' + 1$ and removing this number of segments from a path of length $L'$ is clearly nonsensical.

Also, given a $t$-dimensional vector $\boldsymbol{u} = (u_1, u_2, \dots, u_t)$, we define $(t-1)$-dimensional vectors $\boldsymbol{Q}(\boldsymbol{u}) = (Q_1, Q_2, \dots, Q_{t-1})$ and $\boldsymbol{R}(\boldsymbol{u}) = (R_1, R_2, \dots, R_{t-1})$ as follows[18]:

$$(15) \qquad \begin{aligned} Q_i &= (u_{i+1} + u_{i+3} + u_{i+5} + \cdots) \bmod 2 & 1 \le i < t; \\ R_i &= (t - i + u_{i+1} + u_{i+3} + u_{i+5} + \cdots) \bmod 2 & 1 \le i < t, \end{aligned}$$

where $u_i = 0$ for $i > t$.[19] In the expressions, obtained below, we require summations over vectors $\boldsymbol{m} = (m_1, m_2, \dots, m_{t-1})$ for which $m_i \equiv Q_i \,(\mathrm{mod}\,2)$ for $1 \le i < t$. We shall denote such a restriction on $\boldsymbol{m}$ by simply $\boldsymbol{m} \equiv \boldsymbol{Q}(\boldsymbol{u})$.

To illustrate these definitions, let $p' = 14$, $a = 4$ and $b = 8$. Then $\boldsymbol{u}_{4,8} = (0,0,1,0,0,0,1,0,0,0,0,0)$, $\boldsymbol{Q}(\boldsymbol{u}_{4,8}) = (0,0,0,1,0,1,0,0,0,0,0)$ and $\boldsymbol{R}(\boldsymbol{u}_{4,8}) = (1,0,1,1,1,1,1,0,1,0,1)$.

**4.1. First system.** In this section, we consider a sequence of $\mathcal{B}$-transforms governed by the following values:

$$(16) \qquad \begin{aligned} e_i &= 1 && \text{for} && 1 \le i < a; \\ e_i &= 0 && \text{for} && a \le i \le p' - 1; \\ f_i &= 1 && \text{for} && 1 \le i < b; \\ f_i &= 0 && \text{for} && b \le i \le p' - 1. \end{aligned}$$

It will be useful to define:

$$\begin{aligned} a_j &= a - \sum_{i=1}^{j-1} e_i = \begin{cases} 1 & \text{if } a \le j \le p' - 1, \\ a - j + 1 & \text{if } 1 \le j \le a \end{cases} \\[2mm] b_j &= b - \sum_{i=1}^{j-1} f_i = \begin{cases} 1 & \text{if } b \le j \le p' - 1, \\ b - j + 1 & \text{if } 1 \le j \le b \end{cases} \end{aligned}$$

In fact, although we make no use of this, these values actually give the start and endpoints of the paths in the sequence of sets of paths that we generate.

THEOREM 4.1. *Let* $1 \le a, b < p'$ *and* $c = b - 1$. *Then, with* $\boldsymbol{C} = \boldsymbol{C}^{(p'-2)}$, $m_0 = L$, *and* $m_{p'-2} = 0$,

$$\chi_{a,b,c}^{1,p'}(L)$$
$$= q^{\frac{1}{4}(a-b-L^2)} \sum q^{\frac{1}{4}\tilde{\boldsymbol{m}}\boldsymbol{C}\tilde{\boldsymbol{m}}^T - \frac{1}{2}m_{b-1}} \prod_{i=1}^{p'-3} \begin{bmatrix} \frac{1}{2}(m_{i-1} + m_{i+1} + \delta_{i,a-1} + \delta_{i,b-1}) \\ m_i \end{bmatrix}_q ,$$

*where the above sum is over all* $\boldsymbol{m} = (m_1, m_2, \dots, m_{p'-3})$ *for which* $\boldsymbol{m} \equiv \boldsymbol{Q}(\boldsymbol{u}_{a,b})$, *with* $\tilde{\boldsymbol{m}} = (m_0, m_1, m_2, \dots, m_{p'-3})$.

*Proof:* Let $t = p' - 2$. Lemma 3.7 implies that:[20]

$$\chi_{a_i,b_i,e_i,f_i}^{1,p'+1-i}(m_{i-1}, m_i)$$
$$= \sum_{m \equiv m_{i-1}} q^{\frac{1}{4}\left((m_{i-1}-m_i)^2 - \delta_{e_i+f_i,1}\right)} \begin{bmatrix} \frac{1}{2}(m_{i-1} + m) \\ m_i \end{bmatrix}_q \chi_{a_{i+1},b_{i+1},e_i,f_i}^{1,p'-i}(m_i, m)$$

---

[18]What we denote here as $\boldsymbol{Q}(\boldsymbol{u}_{a,b})$, is denoted $\tilde{\boldsymbol{Q}}_{p'-a,p'-b}$ in (3.3) of [**20**], and as $\boldsymbol{Q}_{p'-a,p'-b}$ in (2.2) of [**25**]. What we denote here as $\boldsymbol{R}(\boldsymbol{u}_{a,b})$, is denoted $\boldsymbol{Q}_{a+1,p'-b}$ in (3.3) of [**20**], and as $\boldsymbol{R}_{a,p'-b}$ in (2.5) of [**25**].

[19]Note that $\boldsymbol{R}(\boldsymbol{u}) = \boldsymbol{Q}(\boldsymbol{u} + \boldsymbol{e}_t)$ where the $t$-dimensional $\boldsymbol{e}_t = (0,0,\dots,0,1)$.

[20]In this and all subsequent proofs, we take the symbol '$\equiv$' to mean equivalence modulo 2.

for $i = 1, 2, \ldots, t$. In the $i$th case, we replace the summation variable $m$ with $m = m_{i+1} + \delta_{i,a-1} + \delta_{i,b-1}$. Thereupon, $m_{i-2} \equiv m_i + \delta_{i,a} + \delta_{i,b}$ for $2 \le i \le t+1$. We now express $\chi^{1,p'-i}_{a_{i+1},b_{i+1},e_i,f_i}(m_i, m)$ in terms of $\chi^{1,p'-i}_{a_{i+1},b_{i+1},e_{i+1},f_{i+1}}(m_i, m_{i+1})$. When $i \ne a - 1$ and $i \ne b - 1$, we immediately obtain

$$\chi^{1,p'-i}_{a_{i+1},b_{i+1},e_i,f_i}(m_i, m_{i+1}) = \chi^{1,p'-i}_{a_{i+1},b_{i+1},e_{i+1},f_{i+1}}(m_i, m_{i+1}),$$

from (16). When $i = a - 1$, so that $a_{i+1} = 1$, $e_i = 1$ and $e_{i+1} = 0$, Lemma 2.9 yields:

$$\chi^{1,p'-i}_{a_{i+1},b_{i+1},e_i,f_i}(m_i, m_{i+1} + 1 + \delta_{a,b}) = \chi^{1,p'-i}_{a_{i+1},b_{i+1},e_{i+1},f_{i+1}+\delta_{a,b}}(m_i, m_{i+1} + \delta_{a,b}),$$

and when $i = b - 1$, so that $b_{i+1} = 1$, $f_i = 1$ and $f_{i+1} = 0$, Lemma 2.9 yields:

$$\chi^{1,p'-i}_{a_{i+1},b_{i+1},e_i-\delta_{a,b},f_i}(m_i, m_{i+1} + 1) = q^{\frac{1}{2}(a_b-1-m_{b-1})}\chi^{1,p'-i}_{a_{i+1},b_{i+1},e_{i+1},f_{i+1}}(m_i, m_{i+1}).$$

By Lemma 2.10, $\chi^{1,2}_{1,1,0,0}(m_t, m_{t+1}) = \delta_{m_t,0}\delta_{m_{t+1},0}$, so that we require $m_t = m_{t+1} = 0$ for a non-zero contribution, whereupon the Gaussian polynomial in the $t$th summation is equal to 1. Moreover, it then follows from $m_{i-2} \equiv m_i + \delta_{i,a} + \delta_{i,b}$ that $(m_1, m_2, m_3, \ldots, m_{t-1}) \equiv \boldsymbol{Q}(\boldsymbol{u}_{a,b})$. We now calculate $\frac{1}{2}(a_b - 1) - \frac{1}{4}\sum_{i=1}^{t}\delta_{e_i+f_i,1} = \frac{1}{4}(a - b)$, by using $e_i + f_i = 1$ in $|a - b|$ cases and $a_b = 1$ if $b \ge a$ and $a_b = a - b + 1$ if $b \le a$. Combining all the above yields:

$$\chi^{1,p'}_{a_1,b_1,e_1,f_1}(m_0, m_1)$$
$$= \sum q^{\frac{1}{4}\left(\sum_{i=1}^{t}(m_{i-1}-m_i)^2 + a - b\right) - \frac{1}{2}m_{b-1}} \prod_{i=1}^{t-1} \begin{bmatrix} \frac{1}{2}(m_{i-1} + m_{i+1} + \delta_{i,a-1} + \delta_{i,b-1}) \\ m_i \end{bmatrix}_q,$$

where $m_t = 0$ and the sum is over all $(m_2, m_3, \ldots, m_{t-1}) \equiv (Q_2, Q_3, \ldots, Q_{t-1})$, when $\boldsymbol{Q}(\boldsymbol{u}_{a,b}) = (Q_1, Q_2, \ldots, Q_{t-1})$. From $m_1 \equiv Q_1$, we obtain $m_0 + e_1 + f_1 \equiv Q_1$. The theorem now follows after noting that

$$\sum_{i=1}^{t}(m_{i-1} - m_i)^2 = \tilde{\boldsymbol{m}}\boldsymbol{C}^{(t)}\tilde{\boldsymbol{m}}^T - L^2,$$

and using Lemma 2.11 in the form:

$$\chi^{1,p'}_{a,b,c}(m_0) = \sum_{m_1 \equiv Q_1} \chi^{1,p'}_{a_1,b_1,e_1,f_1}(m_0, m_1),$$

and noting that $a = a_1$ and $b = b_1$. □

A further expression is obtained by using the reflection symmetry identified in Lemma 2.2:

COROLLARY 4.2. *Let* $1 \le a, b < p'$ *and* $c = b + 1$. *Then, with* $\boldsymbol{C} = \boldsymbol{C}^{(p'-2)}$, $m_0 = L$, *and* $m_{p'-2} = 0$,

$$\chi^{1,p'}_{a,b,c}(L) = q^{\frac{1}{4}(b-a-L^2)}$$
$$\times \sum q^{\frac{1}{4}\tilde{\boldsymbol{m}}\boldsymbol{C}\tilde{\boldsymbol{m}}^T - \frac{1}{2}m_{p'-b-1}} \prod_{i=1}^{p'-3} \begin{bmatrix} \frac{1}{2}(m_{i-1} + m_{i+1} + \delta_{i,p'-a-1} + \delta_{i,p'-b-1}) \\ m_i \end{bmatrix}_q,$$

*where the sum is over all* $\boldsymbol{m} = (m_1, m_2, \ldots, m_{p'-3})$ *for which* $\boldsymbol{m} \equiv \boldsymbol{Q}(\boldsymbol{u}_{p'-a,p'-b})$, *with* $\tilde{\boldsymbol{m}} = (m_0, m_1, m_2, \ldots, m_{p'-3})$.

**4.2. Second system.** In this section, we consider a sequence of $\mathcal{B}$-transforms different to that used in the previous section. This leads to a constant-sign expression for $\chi^{1,p'}_{a,b,b-1}(L)$ that differs from that obtained in Theorem 4.1.

In this section we use:

(17)
$$
\begin{aligned}
e_i &= 0 \quad \text{for} \quad 1 \le i < p' - a; \\
e_i &= 1 \quad \text{for} \quad p' - a \le i \le p' - 1; \\
f_i &= 1 \quad \text{for} \quad 1 \le i < b; \\
f_i &= 0 \quad \text{for} \quad b \le i \le p' - 1.
\end{aligned}
$$

We define:

$$
a_j \;=\; a - \sum_{i=1}^{j-1} e_i = \begin{cases} p' - j & \text{if } p' - a \le j \le p' - 1, \\ a & \text{if } 1 \le j \le p' - a \end{cases}
$$

$$
b_j \;=\; b - \sum_{i=1}^{j-1} f_i = \begin{cases} 1 & \text{if } b \le j \le p' - 1, \\ b - j + 1 & \text{if } 1 \le j \le b \end{cases}
$$

THEOREM 4.3. *Let* $1 \le a, b < p'$ *and* $c = b - 1$. *Then, with* $\mathbf{C} = \mathbf{C}^{(p'-2)}$, $m_0 = L$, *and* $m_{p'-2} = 0$,

$$
\chi^{1,p'}_{a,b,c}(L)
$$
$$
= q^{\frac{1}{4}(a-b-L^2)} \sum q^{\frac{1}{4}\tilde{\mathbf{m}}\mathbf{C}\tilde{\mathbf{m}}^T - \frac{1}{2}m_{b-1}} \prod_{i=1}^{p'-3} \begin{bmatrix} \frac{1}{2}(m_{i-1} + m_{i+1} + \delta_{i,p'-a-1} + \delta_{i,b-1}) \\ m_i \end{bmatrix}_q,
$$

*where the sum is over all* $\mathbf{m} = (m_1, m_2, \ldots, m_{p'-3})$ *for which* $\mathbf{m} \equiv \mathbf{R}(\mathbf{u}_{p'-a,b})$, *with* $\tilde{\mathbf{m}} = (m_0, m_1, m_2, \ldots, m_{p'-3})$.

*Proof:* We proceed much as in the proof of Theorem 4.1. Let $t = p' - 2$. Lemma 3.7 implies that:

$$
\chi^{1,p'+1-i}_{a_i,b_i,e_i,f_i}(m_{i-1}, m_i)
$$
$$
= \sum_{m \equiv m_{i-1}} q^{\frac{1}{4}\left((m_{i-1}-m_i)^2 - \delta_{e_i+f_i,1}\right)} \begin{bmatrix} \frac{1}{2}(m_{i-1} + m) \\ m_i \end{bmatrix}_q \chi^{1,p'-i}_{a_{i+1},b_{i+1},e_i,f_i}(m_i, m)
$$

for $i = 1, 2, \ldots, t$. In the $i$th case, we replace the summation variable $m$ with $m = m_{i+1} + \delta_{i,p'-a-1} + \delta_{i,b-1}$. Thereupon, $m_{i-2} \equiv m_i + \delta_{i,p'-a} + \delta_{i,b}$ for $2 \le i \le t+1$. When $i \ne p' - a - 1$ and $i \ne b - 1$, we immediately obtain

$$
\chi^{1,p'-i}_{a_{i+1},b_{i+1},e_i,f_i}(m_i, m_{i+1}) = \chi^{1,p'-i}_{a_{i+1},b_{i+1},e_{i+1},f_{i+1}}(m_i, m_{i+1}),
$$

When $i = p' - a - 1$, so that $a_{i+1} = p' - i - 1$, $e_i = 0$ and $e_{i+1} = 1$, Lemma 2.9 yields:

$$
\chi^{1,p'-i}_{a_{i+1},b_{i+1},e_i,f_i}(m_i, m_{i+1}+1+\delta_{a+b,p'}) = \chi^{1,p'-i}_{a_{i+1},b_{i+1},e_{i+1},f_{i+1}+\delta_{a+b,p'}}(m_i, m_{i+1}+\delta_{a+b,p'}),
$$

and when $i = b - 1$, so that $b_{i+1} = 1$, $f_i = 1$ and $f_{i+1} = 0$, Lemma 2.9 yields:

$$
\chi^{1,p'-i}_{a_{i+1},b_{i+1},e_i+\delta_{a+b,p'},f_i}(m_i, m_{i+1} + 1) = q^{\frac{1}{2}(a_b-1-m_{b-1})} \chi^{1,p'-i}_{a_{i+1},b_{i+1},e_{i+1},f_{i+1}}(m_i, m_{i+1}).
$$

By Lemma 2.10, $\chi^{1,2}_{1,1,1,0}(m_t, m_{t+1}) = \delta_{m_t,0}\delta_{m_{t+1},1}$, so that we require $m_t = 0$ and $m_{t+1} = 1$ for a non-zero contribution, whereupon the Gaussian polynomial in the $t$th summation is equal to 1. Moreover, it then follows from $m_{i-2} \equiv m_i + \delta_{i,p'-a} +$

$\delta_{i,b}$ that $(m_1, m_2, m_3, \ldots, m_{t-1}) \equiv \boldsymbol{R}(\boldsymbol{u}_{p'-a,b})$. We now calculate $\frac{1}{2}(a_b - 1) - \frac{1}{4}\sum_{i=1}^{t}\delta_{e_i+f_i,1} = \frac{1}{4}(a-b)$, by using $e_i + f_i = 1$ in $t - |t+2-a-b|$ cases and $a_b = a$ if $b \le t - a + 2$ and $a_b = t - b + 2$ if $b \ge t - a + 2$. Combining all the above yields:

$$\chi_{a_1,b_1,e_1,f_1}^{1,p'}(m_0, m_1)$$
$$= \sum q^{\frac{1}{4}\left(\sum_{i=1}^{t}(m_{i-1}-m_i)^2+a-b\right)-\frac{1}{2}m_{b-1}}\prod_{i=1}^{t-1}\begin{bmatrix}\frac{1}{2}(m_{i-1}+m_{i+1}+\delta_{i,p'-a-1}+\delta_{i,b-1})\\m_i\end{bmatrix}_q,$$

where $m_t = 0$ and the sum is over all $(m_2, m_3, \ldots, m_{t-1}) \equiv (R_2, R_3, \ldots, R_{t-1})$ when $\boldsymbol{R}(\boldsymbol{u}_{p'-a,b}) = (R_1, R_2, \ldots, R_{t-1})$. From $m_1 \equiv R_1$, we obtain $m_0 + e_1 + f_1 \equiv R_1$. The theorem now follows after noting that

$$\sum_{i=1}^{t}(m_{i-1}-m_i)^2 = \tilde{\boldsymbol{m}}\boldsymbol{C}^{(t)}\tilde{\boldsymbol{m}}^T - L^2,$$

and using Lemma 2.11 in the form:

$$\chi_{a,b,c}^{1,p'}(m_0) = \sum_{m_1 \equiv R_1} \chi_{a_1,b_1,e_1,f_1}^{1,p'}(m_0, m_1),$$

and noting that $a = a_1$ and $b = b_1$. $\qquad\qquad\square$

Again, we use Lemma 2.2 to obtain a further expression:

COROLLARY 4.4. *Let* $1 \le a,b < p'$ *and* $c = b + 1$. *Then, with* $\boldsymbol{C} = \boldsymbol{C}^{(p'-2)}$, $m_0 = L$, *and* $m_{p'-2} = 0$,

$$\chi_{a,b,c}^{1,p'}(L)$$
$$= q^{\frac{1}{4}(b-a-L^2)}\sum q^{\frac{1}{4}\tilde{\boldsymbol{m}}\boldsymbol{C}\tilde{\boldsymbol{m}}^T-\frac{1}{2}m_{p'-b-1}}\prod_{i=1}^{p'-3}\begin{bmatrix}\frac{1}{2}(m_{i-1}+m_{i+1}+\delta_{i,a-1}+\delta_{i,p'-b-1})\\m_i\end{bmatrix}_q,$$

*where the sum is over all* $\boldsymbol{m} = (m_1, m_2, \ldots, m_{p'-3})$ *for which* $\boldsymbol{m} \equiv \boldsymbol{R}(\boldsymbol{u}_{a,p'-b})$, *with* $\tilde{\boldsymbol{m}} = (m_0, m_1, m_2, \ldots, m_{p'-3})$.

## 5. ABF generating functions

We now use Lemma 2.4 to convert the constant-sign expressions obtained above to the ABF case.

THEOREM 5.1. *Let* $1 \le a,b < p'$, $\boldsymbol{C} = \boldsymbol{C}^{(p'-3)}$, $m_0 = L$ *and* $m_{p'-2} = 0$. *Then if* $b > 1$ *and* $c = b - 1$,

$$\chi_{a,b,c}^{p'-1,p'}(L)$$
$$= f_{a,b,c}\sum_{\boldsymbol{m}\equiv\boldsymbol{Q}(\boldsymbol{u}_{a,b})} q^{\frac{1}{4}\boldsymbol{m}\boldsymbol{C}\boldsymbol{m}^T-\frac{1}{2}m'_{a-1}}\prod_{i=1}^{p'-3}\begin{bmatrix}\frac{1}{2}(m_{i-1}+m_{i+1}+\delta_{i,a-1}+\delta_{i,b-1})\\m_i\end{bmatrix}_q,$$

*and also*

$$\chi_{a,b,c}^{p'-1,p'}(L)$$
$$= f_{a,b,c}\sum_{\boldsymbol{m}\equiv\boldsymbol{R}(\boldsymbol{u}_{p'-a,b})} q^{\frac{1}{4}\boldsymbol{m}\boldsymbol{C}\boldsymbol{m}^T-\frac{1}{2}m'_{p'-a-1}}\prod_{i=1}^{p'-3}\begin{bmatrix}\frac{1}{2}(m_{i-1}+m_{i+1}+\delta_{i,p'-a-1}+\delta_{i,b-1})\\m_i\end{bmatrix}_q,$$

*and if $b < p' - 1$ and $c = b + 1$ then:*

$$\chi_{a,b,c}^{p'-1,p'}(L)$$
$$= f_{a,b,c} \sum_{\boldsymbol{m} \equiv \boldsymbol{Q}(\boldsymbol{u}_{p'-a,p'-b})} q^{\frac{1}{4}\boldsymbol{mCm}^T - \frac{1}{2}m'_{p'-a-1}} \prod_{i=1}^{p'-3} \begin{bmatrix} \frac{1}{2}(m_{i-1} + m_{i+1} + \delta_{i,p'-a-1} + \delta_{i,p'-b-1}) \\ m_i \end{bmatrix}_q,$$

*and also*

$$\chi_{a,b,c}^{p'-1,p'}(L)$$
$$= f_{a,b,c} \sum_{\boldsymbol{m} \equiv \boldsymbol{R}(\boldsymbol{u}_{a,p'-b})} q^{\frac{1}{4}\boldsymbol{mCm}^T - \frac{1}{2}m'_{a-1}} \prod_{i=1}^{p'-3} \begin{bmatrix} \frac{1}{2}(m_{i-1} + m_{i+1} + \delta_{i,a-1} + \delta_{i,p'-b-1}) \\ m_i \end{bmatrix}_q,$$

*where $f_{a,b,c} = q^{-\frac{1}{4}(a-b)(a-c)}$, and in each case $m'_i = m_i$ for $i > 0$ and $m'_0 = 0$.*

*Proof:* Using $\begin{bmatrix} P \\ Q \end{bmatrix}_{q^{-1}} = q^{-Q(P-Q)} \begin{bmatrix} P \\ Q \end{bmatrix}_q$ yields:

$$\prod_{i=1}^{p'-3} \begin{bmatrix} \frac{1}{2}(m_{i-1} + m_{i+1} + \delta_{i,a-1} + \delta_{i,b-1}) \\ m_i \end{bmatrix}_{q^{-1}}$$
$$= \prod_{i=1}^{p'-3} q^{-\frac{1}{2}m_i(m_{i-1} - 2m_i + m_{i+1} + \delta_{i,a-1} + \delta_{i,b-1})} \begin{bmatrix} \frac{1}{2}(m_{i-1} + m_{i+1} + \delta_{i,a-1} + \delta_{i,b-1}) \\ m_i \end{bmatrix}_q$$
$$= q^{\frac{1}{2}(\boldsymbol{mCm}^T - m_1 m_0 - m'_{a-1} - m'_{b-1})} \prod_{i=1}^{p'-3} \begin{bmatrix} \frac{1}{2}(m_{i-1} + m_{i+1} + \delta_{i,a-1} + \delta_{i,b-1}) \\ m_i \end{bmatrix}_q.$$

Substituting this and $\tilde{\boldsymbol{m}}\boldsymbol{C}^{(p'-2)}\tilde{\boldsymbol{m}}^T = \boldsymbol{mCm}^T + 2L^2 - 2m_0 m_1$ into Theorem 4.1, and noting that $b > 1$, we obtain:

$$\chi_{a,b,c}^{1,p'}(L; q^{-1})$$
$$= q^{\frac{1}{4}(b-a-L^2)} \sum q^{\frac{1}{4}\boldsymbol{mCm}^T - \frac{1}{2}m'_{a-1}} \prod_{i=1}^{p'-3} \begin{bmatrix} \frac{1}{2}(m_{i-1} + m_{i+1} + \delta_{i,a-1} + \delta_{i,b-1}) \\ m_i \end{bmatrix}_q.$$

The first expression then follows from Lemma 2.4. The other three expressions arise in a similar way from Theorem 4.3, Corollary 4.2 and Corollary 4.4 respectively. □

## Appendix A. $mn$-systems

Consider once more the proof of Lemma 3.7. There, It is shown that for each path $h' \in \mathcal{P}_{a,b,e,f}^{p'}(L, m)$, there is a unique pair $(h, k)$ with $h \in \mathcal{P}_{a-e,b-f,e,f}^{p'-1}(m', m)$ and $k \in \mathbb{N}$, for which $h'$ arises from the action of a $\mathcal{B}$-transform on $h$, followed by the insertion of $k$ particles, followed by moving these particles in some way. The path $h$ will be referred to as the $(e, f)$-*antecedent* of $h'$. The value of $k$ will be referred to as the $(e, f)$-*particle content* of $h'$. From the proof of Lemma 3.7, we see that it is given by:

$$(18) \qquad 2k = L' - 2m' + m.$$

Now, given $e_i, f_i \in \{0, 1\}$ for $1 \le i \le t = p' - 2$, we may iterate the above procedure. Let $h \in \mathcal{P}_{a,b,e,f}^{p'}(L, m)$, and let $n_1$ be its $(e_1, f_1)$-particle content and $h'$ its $(e_1, f_1)$-antecedent (we deviate from the above priming convention). Now

let $n_2$ and $h''$ be respectively the $(e_2, f_2)$-particle content and $(e_2, f_2)$-antecedent of $h'$. Proceeding in this way, we obtain a vector $\boldsymbol{n} = (n_1, n_2, \ldots, n_t)$, which we shall simply refer to as the particle content of $h$. Note that the particle content is dependent on the particular sequence of $e_i, f_i \in \{0, 1\}$ being considered. Thus, in general, a given path $h$ has differing particle contents in the two systems considered in Sections 4.1 and 4.2.

**A.1. First system.** On examining the proof of Theorem 4.1, we see that the use of the $i$th $\mathcal{B}$-transform therein, results from substituting $L' = m_{i-1}$, $m' = m_i$ and $m = m_{i+1} + \delta_{i,a-1} + \delta_{i,b-1}$ into Lemma 3.7. Therefore, with $e_i$ and $f_i$ given by (16), we find that if a path $h \in \mathcal{P}_{a,b,c}^{p'}(L)$ has particle content $(n_1, n_2, \ldots, n_t)$ then, from (18),

$$(19) \qquad m_{j-1} + m_{j+1} = 2m_j + 2n_j - \delta_{j,a-1} - \delta_{j,b-1},$$

for $1 \le j \le t$, where $m_0 = L$ and $m_t = m_{t+1} = 0$.

The set of $t$ equations given by (19) defines an interdependence between the vectors $\boldsymbol{n} = (n_1, n_2, \ldots, n_t)$ and $\boldsymbol{m} = (m_0, m_1, \ldots, m_{t-1})$ known as the $\boldsymbol{mn}$-system.

If we define the vector $\boldsymbol{u}_{a,b} = (u_1, u_2, \ldots, u_t)$ with components $u_j = \delta_{j,a-1} + \delta_{j,b-1}$, the $\boldsymbol{mn}$-system described in this section may be conveniently written:

$$(20) \qquad 2\boldsymbol{n} = -\boldsymbol{m}\boldsymbol{C}^{(t)} + \boldsymbol{u}_{a,b},$$

where $\boldsymbol{C}^{(t)}$ is the Cartan matrix of type $A_t$, defined by (13).

**A.2. Second system.** By the same means as in Section A.1, we obtain the $\boldsymbol{mn}$-system for the case considered in Section 4.2. Thus, with $e_i, f_i$ defined by (17), the proof of Theorem 4.3 shows that if a path $h \in \mathcal{P}_{a,b,c}^{p'}(L)$ has particle content $(n_1, n_2, \ldots, n_t)$ then

$$(21) \qquad m_{j-1} + m_{j+1} = 2m_j + 2n_j - \delta_{j,p'-a-1} - \delta_{j,b-1} - \delta_{j,t},$$

for $1 \le j \le t$, where $m_0 = L$ and $m_t = m_{t+1} = 0$. We then obtain:

$$(22) \qquad 2\boldsymbol{n} = -\boldsymbol{m}\boldsymbol{C}^{(t)} + \boldsymbol{u}_{p'-a,b} + \boldsymbol{e}_t,$$

where the $t$-dimensional $\boldsymbol{e}_t = (0, 0, \ldots, 0, 1)$.

## Appendix B. Bijection between paths and partitions

In this appendix, we briefly describe a natural weight-preserving bijection between the paths that we've been considering in this paper and partitions that satisfy certain hook-difference conditions [**2, 15**]. Such a bijection occurs in both the parafermion and the ABF cases. In fact, as pointed out in [**2**], these bijections are just special cases of a more general bijection existing for the weighted paths of [**9**]. We give a full description (of the general case) in [**15**].

**B.1. Partitions with prescribed hook-difference conditions.** A partition $\mu = (\mu_1, \mu_2, \ldots, \mu_M)$ is a sequence of $M$ integer parts $\mu_1, \mu_2, \ldots, \mu_M$ satisfying $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_M \ge 0$. The weight $\text{wt}(\mu)$ of $\mu$ is given by $\text{wt}(\mu) = \sum_{i=1}^{M} \mu_i$. The partition $\mu$ is often depicted by its *Young diagram* (also called Ferrars graph), $F^\mu$ which comprises $M$ left-adjusted rows, the $i$th row of which (reading down) consists of $\mu_i$ cells [**4**]. The coordinate $(i, j)$ of a cell is obtained by setting $i$ and $j$ to be respectively, the row and column (reading from the left) in which the cell

resides. The $k$th diagonal of $F^\mu$ comprises all those cells of $F^\mu$ with coordinates $(i, j)$ which satisfy $i - j = k$.

The partition $\mu'$, conjugate to $\mu$, is obtained by setting $\mu'_j$ to be the number of cells in the $j$th column of $F^\mu$. The hook-difference at the cell with coordinate $(i, j)$ is then defined to be $\mu_i - \mu'_j$. As an example, filling each cell of $F^{(5,4,3,1)}$ with its hook difference, yields:

| 1 | **2** | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 1 | **1** | 2 | |
| −1 | 0 | 0 | | |
| −3 | | | | |

.

The bold entries are those on diagonal $-1$. In what follows, we will be especially interested in the hook-differences on certain diagonals.

Let $K, i, N, M, \alpha, \beta$ be non-negative integers for which $1 \le i \le K/2$, $\alpha + \beta < K$ and $\beta - i \le N - M \le K - \alpha - i$. In [2], $D_{K,i}(N, M; \alpha, \beta)$ is defined to be the generating function for partitions $\mu$ into at most $M$ parts, each not exceeding $N$ such that the hook differences on diagonal $1 - \beta$ are at least $\beta - i + 1$, and on diagonal $\alpha - 1$ are at most $K - i - \alpha - 1$. In addition, if $\alpha = 0$, the restriction that $\mu_{N-L+i+1} > 0$ is also imposed; and if $\beta = 0$, the restriction that $\mu_1 > M - i$ is also imposed.

**B.2. Parafermions.** In the parafermion case, the bijective image of a path $h \in \mathcal{P}^{p'}_{a,b,c}(L)$ is obtained as follows. Begin with an empty Young diagram. Now traverse the path from left to right. If a scoring vertex that contributes $x$ to the weight is encountered, append a new first row of length $x$ to the top of the Young diagram. If a scoring vertex that contributes $y$ to the weight is encountered, append a new first column of length $y$ to the left of the Young diagram. The diagram is not changed at non-scoring vertices. In this way, after all $L$ vertices have been considered, a Young diagram $F^\mu$ results.

For example, consider the path shown in Fig. 1, and let $c = b + 1$. Here, we obtain the following Young diagram:
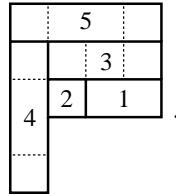


Here, the entries indicate the order in which the pieces have been added.

If $c = b + 1$ let $r = 0$, and if $c = b - 1$ let $r = 1$. It may be shown ([15], or by using the techniques of [17]) that each hook-difference on diagonal $1 - r$ of $\mu$ is at least $r - a + 1$, and each hook-difference on diagonal $-r$ of $\mu$ is at most $r - a + p'$. Moreover, it may be shown that the map is in fact a bijection. Since clearly $\mathrm{wt}(\mu) = \mathrm{wt}(h)$, we have

$$\chi^{1,p'}_{a,b,c}(L) = D_{p',a}\left(\frac{L - a + b}{2}, \frac{L + a - b}{2}, 1 - r, r\right).$$

**B.3. ABF.** The description of the bijection in the ABF case proceeds similarly. We must first provide an analogue of $c$ that was defined in Section 2.4. If the $i$th vertex has coordinates $(x, y)$, define $\tilde{c}(h_{i-1}, h_i, h_{i+1})$ as follows:

$$
\begin{aligned}
\tilde{c}(h - 1, h, h + 1) &= x\,; \\
\tilde{c}(h + 1, h, h - 1) &= y\,; \\
\tilde{c}(h - 1, h, h - 1) &= 0\,; \\
\tilde{c}(h + 1, h, h + 1) &= 0\,.
\end{aligned}
$$

Now, if we define

(23)
$$
\mathrm{wt}^{\,\mathrm{III}}(h) = \sum_{i=1}^{L} \tilde{c}_i(h_{i-1}, h_i, h_{i+1}),
$$

then, as is readily shown,

(24)
$$
\chi_{a,b,c}^{p'-1,p'}(L) = \sum_{h \in \mathcal{P}_{a,b,c}^{p'}(L)} q^{\mathrm{wt}^{\,\mathrm{III}}(h)},
$$

where $\chi_{a,b,c}^{p'-1,p'}(L)$ is given by (3) and (2).

The partition $\mu$ is now obtained exactly as in the above description of the parafermion case: the only difference being that the non-scoring vertices there are scoring vertices here, and vice-versa.

For example, again consider the path shown in Fig. 1, and let $c = b + 1$. In the ABF case, we obtain the following Young diagram:



.
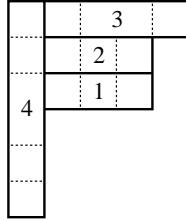
Let $r = \min(b, c)$. It may be shown that each hook-difference on diagonal $1 - r$ of $\mu$ is at least $r - a + 1$, and each hook-difference on diagonal $p' - 2 - r$ of $\mu$ is at most $r - a + p'$. Once more, the map may be shown to be a bijection, whereupon:

$$
\chi_{a,b,c}^{p'-1,p'}(L) = D_{p',a}\left(\frac{L - a + b}{2}, \frac{L + a - b}{2}, p' - r - 1, r\right).
$$

(cf. eq. (5.1) of [**2**]).

Finally, we note that this bijection generalises that given in [**17**], and that the method given there may be readily extended to deal with the current case.

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Department of Mathematics and Statistics, The University of Melbourne, Parkville, Victoria 3052, Australia.
*E-mail address*: `foda@maths.mu.oz.au`

Department of Mathematics and Statistics, The University of Melbourne, Parkville, Victoria 3052, Australia.
*E-mail address*: `trevor@maths.mu.oz.au`